Fourier Series

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Contents

1 PERIODIC FUNCTIONS

Fourier series are infinite series that represent periodic functions in terms of cosines and sines. Solutions of many problems in engineering and applied mathematics can be approximated using Fourier series. They include problems of vibrations of a string and heat conduction problems. In this unit, we introduce the Fourier series, the conditions of existence of it, applications and some problems solved using Fourier series. We begin with the definition of periodic functions.

Definition 1.1

A function $f(x)$ is called a periodic function *if there is some positive number p such that*

$$f(x+p) = f(x) \tag{1}$$

for every x. The number p is called a period *of f.*

Examples

1. The period of $\sin x$ is 2π.
2. The period of $\sin 2\pi x$ is 1.
3. The period of $\sin \pi x$ is 2.
4. Generally the period of $\sin lx$ is $\frac{2\pi}{l}$, for $l \neq 0$.
5. Also the period of $\sin(lx + a)$ is $\frac{2\pi}{l}$, for $l \neq 0$.
6. The period of $\cos x$ is 2π.
7. The period of a simple harmonic motion with displacement $y = \sin \omega t$ is $\frac{2\pi}{\omega}$.
8. The period of $s = 2\sin(4t - 1)$ is $\frac{2\pi}{4} = \frac{\pi}{2}$.
9. The period of $s = 2\sin 3t \cos 3t$ is $\frac{2\pi}{6} = \frac{\pi}{3}$ since $2\sin 3t \cos 3t = \sin 6t$
10. The period of $z = \mathrm{e}^{\mathrm{i}t}$ is 2π, since $\mathrm{e}^{\mathrm{i}2\pi} = \cos 2\pi + \mathrm{i}\sin 2\pi = 1$, we have $\mathrm{e}^{\mathrm{i}t} = \mathrm{e}^{\mathrm{i}t}\mathrm{e}^{\mathrm{i}2\pi} = \mathrm{e}^{\mathrm{i}(t+2\pi)}$.
11. The period of $z = 2\mathrm{e}^{\mathrm{i}t/2}$ is $\frac{2\pi}{1/2} = 4\pi$.
12. The period of $z = 2\mathrm{e}^{\mathrm{i}\pi t}$ is $\frac{2\pi}{\pi} = 2$.
13. The period of $z = -4\mathrm{e}^{\mathrm{i}(2t+3\pi)}$ is $\frac{2\pi}{2} = \pi$.

2 FOURIER SERIES

Suppose that $f(x)$ is a given function of period 2π defined in the interval $[-\pi, \pi]$. The *Fourier series* of $f(x)$ is defined as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \tag{2}$$

where the Fourier coefficients a_0, a_n and b_n are given by

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx \tag{3}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \tag{4}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \tag{5}$$

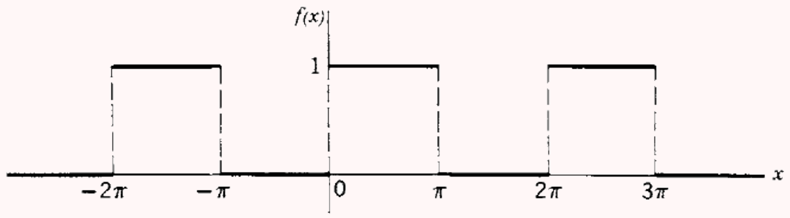
for $n = 1, 2, \ldots$.

▶ Expanding the summation in (2), we have

$$f(x) = \frac{a_0}{2} + a_1\cos x + a_2\cos 2x + a_3\cos 3x + \cdots + b_1\sin x + b_2\sin 2x + b_3\sin 3x + \cdots.$$

Problem 2.1

Expand in a Fourier series the function $f(x)$ sketched in the figure. This function might represent, for example, a periodic voltage pulse. The terms of our Fourier series would then correspond to the different a-c frequencies which are com- bined in this "square wave" voltage, and the magnitude of the Fourier coefficients would indicate the relative importance of the various frequencies.



Solution. From the figure it is clear that $f(x)$ is a period function with period 2π given by,

$$f(x) = \begin{cases} 0, & -\pi \le x \le \pi \\ 1, & 0 \le x \le \pi. \end{cases}$$

The Fourier series of $f(x)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx).$$

We find the Fourier coefficients. By the definition,

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx \\ &= \frac{1}{\pi}\int_{-\pi}^{0} 0\,dx + \int_{0}^{\pi} 1\,dx \\ &= \frac{1}{\pi}\left(0 + [x]_0^{\pi}\right) \\ &= \frac{1}{\pi}(\pi - 0) \\ &= 1. \end{aligned}$$

For $n = 1, 2, \ldots$,

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \\ &= \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\cos nx\,dx + \int_{0}^{\pi} 1\cos nx\,dx\right] \\ &= \frac{1}{\pi}\int_{0}^{\pi} \cos nx\,dx \\ &= \frac{1}{\pi}\left[\frac{\sin nx}{n}\right]_0^{\pi} \\ &= \frac{1}{n\pi}(\sin n\pi - \sin 0) \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \\ &= \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\sin nx\,dx + \int_{0}^{\pi} 1\sin nx\,dx\right] \\ &= \int_{0}^{\pi} \sin nx\,dx \\ &= \frac{1}{\pi}\left[\frac{-\cos nx}{n}\right]_0^{\pi} \\ &= \frac{-1}{n\pi}(\cos n\pi - \cos 0) \\ &= \frac{-1}{n\pi}((-1)^n - 1) \\ &= \frac{1}{n\pi}(1 - (-1)^n) \\ &= \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{2}{n\pi} & \text{if } n \text{ is odd.} \end{cases} \end{aligned}$$

Hence

$$\begin{aligned} f(x) &= \frac{a_0}{2} + a_1\cos x + a_2\cos 2x + a_3\cos 3x + \cdots + b_1\sin x + b_2\sin 2x + b_3\sin 3x + \cdots \\ &= \frac{1}{2} + 0 + \frac{2}{\pi}\sin x + 0 + \frac{2}{3\pi}\sin 3x + 0 + \frac{2}{5\pi}\sin 5x + \cdots \\ &= \frac{1}{2} + \frac{2}{\pi}\left(\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots\right). \end{aligned}$$

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Problem 2.2

Find the Fourier series of

$$f(x) = \begin{cases} -1, & -\pi < x < 0 \\ 1, & 0 < x < \pi. \end{cases}$$

Solution. The Fourier series of $f(x)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx).$$

We find the Fourier coefficients. By the definition,

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx \\ &= \frac{1}{\pi}\left[\int_{-\pi}^{0} -1\,dx + \int_{0}^{\pi} 1\,dx\right] \\ &= \frac{1}{\pi}\left([-x]_{-\pi}^{0} + [x]_0^{\pi}\right) \\ &= \frac{1}{\pi}(0 - \pi + \pi - 0) \\ &= 0. \end{aligned}$$

For $n = 1, 2, \ldots$,

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \\ &= \frac{1}{\pi}\left[\int_{-\pi}^{0}(-1)\cos nx\,dx + \int_{0}^{\pi} 1\cos nx\,dx\right] \\ &= \frac{1}{\pi}\left[-\int_{-\pi}^{0}\cos nx\,dx + \int_{0}^{\pi}\cos nx\,dx\right] \\ &= \frac{1}{\pi}\left[-\frac{\sin nx}{n}\bigg|_{-\pi}^{0} + \frac{\sin nx}{n}\bigg|_{0}^{\pi}\right] \\ &= \frac{1}{n\pi}(\sin 0 + \sin(-n\pi) + \sin n\pi - \sin 0) \\ &= 0 \end{aligned}$$

and

$$\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \\
&= \frac{1}{\pi}\left[\int_{-\pi}^{0}(-1)\sin nx\,dx + \int_{0}^{\pi} 1\sin nx\,dx\right] \\
&= \frac{1}{\pi}\left[\int_{-\pi}^{0} -\sin nx\,dx + \int_{0}^{\pi}\sin nx\,dx\right] \\
&= \frac{1}{\pi}\left[\frac{\cos nx}{n}\bigg|_{-\pi}^{0} + \frac{-\cos nx}{n}\bigg|_{0}^{\pi}\right] \\
&= \frac{1}{n\pi}\left[\cos 0 - \cos(-n\pi) - (\cos n\pi - \cos 0)\right] \\
&= \frac{1}{n\pi}\left[1 - (-1)^n - (-1)^n + 1)\right] \\
&= \frac{1}{n\pi}\left[2 - 2(-1)^n\right] \\
&= \frac{2}{n\pi}(1 - (-1)^n) \\
&= \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{4}{n\pi} & \text{if } n \text{ is odd.} \end{cases}
\end{aligned}$$

Hence

$$\begin{aligned}
f(x) &= \frac{a_0}{2} + a_1\cos x + a_2\cos 2x + a_3\cos 3x + \cdots + b_1\sin x + b_2\sin 2x + b_3\sin 3x + \cdots \\
&= 0 + 0 + \frac{4}{\pi}\sin x + 0 + \frac{4}{3\pi}\sin 3x + 0 + \frac{4}{5\pi}\sin 5x + \cdots \\
&= \frac{4}{\pi}\left(\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots\right).
\end{aligned}$$

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Problem 2.3

Find the Fourier series of

$$f(x) = \begin{cases} 0, & -\pi < x < 0 \\ 1, & 0 < x < \frac{\pi}{2} \\ 0, & \frac{\pi}{2} < x < \pi. \end{cases}$$

Solution. The Fourier series of $f(x)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx).$$

We find the Fourier coefficients. By the definition,

$$\begin{aligned}
a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx \\
&= \frac{1}{\pi}\int_{-\pi}^{0} 0\,dx + \int_{0}^{\frac{\pi}{2}} 1\,dx + \int_{\frac{\pi}{2}}^{\pi} 0\,dx \\
&= \frac{1}{\pi}\int_{0}^{\frac{\pi}{2}} 1\,dx \\
&= \frac{1}{\pi}[x]_0^{\frac{\pi}{2}} \\
&= \frac{1}{\pi}(\frac{\pi}{2} - 0) \\
&= \frac{1}{2}.
\end{aligned}$$

For $n = 1, 2, \ldots,$

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \\
&= \frac{1}{\pi}\int_{-\pi}^{0} 0\,dx + \int_{0}^{\frac{\pi}{2}} 1\cos nx\,dx + \int_{\frac{\pi}{2}}^{\pi} 0\,dx \\
&= \frac{1}{\pi}\int_{0}^{\frac{\pi}{2}} \cos nx\,dx \\
&= \frac{1}{\pi}\left[\frac{\sin nx}{n}\right]_0^{\frac{\pi}{2}} \\
&= \frac{1}{n\pi}\left[\sin n\frac{\pi}{2} - \sin 0\right] \\
&= \frac{1}{n\pi}\sin n\frac{\pi}{2} \\
&= \frac{1}{n\pi}\begin{cases} 1, & n = 1, 5, \ldots \\ 0, & n = 2, 4, \ldots \\ -1, & n = 3, 7, \ldots \end{cases}
\end{aligned}$$

and

$$
\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx \\
&= \frac{1}{\pi}\int_{-\pi}^{0} 0\, dx + \int_{0}^{\frac{\pi}{2}} 1\sin nx\, dx + \int_{\frac{\pi}{2}}^{\pi} 0\, dx \\
&= \frac{1}{\pi}\int_{0}^{\frac{\pi}{2}} \sin nx\, dx \\
&= \frac{1}{\pi}\left[\frac{-\cos nx}{n}\right]_0^{\frac{\pi}{2}} \\
&= \frac{-1}{n\pi}\left[\cos n\frac{\pi}{2} - \cos 0\right] \\
&= \frac{-1}{n\pi}\left[\cos n\frac{\pi}{2} - 1\right] \\
&= \frac{-1}{n\pi}\begin{cases} 0-1, & n = 1,3,5,\ldots \\ -1-1, & n = 2,6,\ldots \\ 1-1, & n = 4,8,\ldots \end{cases} \\
&= \frac{-1}{n\pi}\begin{cases} -1, & n = 1,3,5,\ldots \\ -2, & n = 2,6,\ldots \\ 0, & n = 4,8,\ldots \end{cases} \\
&= \frac{1}{n\pi}\begin{cases} 1, & n = 1,3,5,\ldots \\ 2, & n = 2,6,\ldots \\ 0, & n = 4,8,\ldots \end{cases}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
f(x) &= \frac{a_0}{2} + a_1\cos x + a_2\cos 2x + a_3\cos 3x + \cdots + b_1\sin x + b_2\sin 2x + b_3\sin 3x + \cdots \\
&= \frac{1}{4} + \frac{1}{\pi}\left(\frac{\cos x}{1} - \frac{\cos 3x}{3} + \frac{\cos 5x}{5} - \frac{\cos 7x}{7} + \cdots\right) \\
&\quad + \frac{1}{\pi}\left(\frac{\sin x}{1} + \frac{2\sin 2x}{2} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \frac{2\sin 6x}{6} + \frac{\sin 7x}{7} + \frac{\sin 9x}{9} + \cdots\right).
\end{aligned}
$$

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Problem 2.4

Find the Fourier series of

$$
f(x) = \begin{cases} 0, & -\pi < x < 0 \\ x, & 0 < x < \pi. \end{cases}
$$

Solution. The Fourier series of $f(x)$ is given by

$$
f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx).
$$

We find the Fourier coefficients. By the definition,

$$\begin{aligned}
a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx \\
&= \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\,dx + \int_0^{\pi} x\,dx\right] \\
&= \frac{1}{\pi}\int_0^{\pi} x\,dx \\
&= \frac{1}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi} \\
&= \frac{1}{\pi}\frac{\pi^2}{2} \\
&= \frac{\pi}{2}.
\end{aligned}$$

For $n = 1, 2, \ldots$,

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \\
&= \frac{1}{\pi}\left[\int_{-\pi}^{0} (0)\cos nx\,dx + \int_0^{\pi} x\cos nx\,dx\right] \\
&= \frac{1}{\pi}\int_0^{\pi} x\cos nx\,dx \\
&= \frac{1}{\pi}\left[x\frac{\sin nx}{n}\bigg|_0^{\pi} - \int_0^{\pi}(1)\frac{\sin nx}{n}\,dx\right] \\
&= \frac{1}{\pi}\left[\pi\frac{\sin n\pi}{n} - 0 + \frac{\cos nx}{n^2}\bigg|_0^{\pi}\right] \\
&= \frac{1}{\pi}\left[0 + \frac{\cos n\pi}{n^2} - \frac{\cos 0}{n^2}\right] \\
&= \frac{1}{n^2\pi}[(-1)^n - 1] \\
&= \frac{1}{n^2\pi}\begin{cases} -2, & n = 1, 3, 5, \ldots \\ 0, & n = 2, 4, 6, \ldots. \end{cases}
\end{aligned}$$

and

$$
\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \\
&= \frac{1}{\pi}\left[\int_{-\pi}^{0}(0)\sin nx\,dx + \int_{0}^{\pi} x\sin nx\,dx\right] \\
&= \frac{1}{\pi}\int_{0}^{\pi} x\sin nx\,dx \\
&= \frac{1}{\pi}\left[x\frac{-\cos nx}{n}\bigg|_0^{\pi} - \int_0^{\pi}(1)\frac{-\cos nx}{n}\,dx\right] \\
&= \frac{1}{\pi}\left[\pi\frac{-\cos n\pi}{n} - 0 + \frac{\sin nx}{n^2}\bigg|_0^{\pi}\right] \\
&= \frac{1}{\pi}\left[\pi\frac{(-1)^{n+1}}{n} + \frac{\sin n\pi}{n^2} - \frac{\sin 0}{n^2}\right] \\
&= \frac{1}{\pi}\left[\pi\frac{(-1)^{n+1}}{n} + 0 - 0\right] \\
&= \frac{(-1)^{n+1}}{n}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
f(x) &= \frac{a_0}{2} + a_1\cos x + a_2\cos 2x + a_3\cos 3x + \cdots + b_1\sin x + b_2\sin 2x + b_3\sin 3x + \cdots \\
&= \frac{\pi}{4} - \frac{2}{\pi}\cos x + 0 - \frac{2}{3^2\pi}\cos 3x + 0 - \frac{2}{5^2\pi}\cos 5x + \cdots \\
&\quad + \sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \frac{1}{4}\sin 4x + \cdots \\
&= \frac{\pi}{4} - \frac{2}{\pi}\left(\frac{\cos x}{1} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right) \\
&\quad + \left(\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \cdots\right).
\end{aligned}
$$

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Problem 2.5

Find the Fourier series of

$$
f(x) = \begin{cases} -x, & -\pi < x < 0 \\ x, & 0 < x < \pi. \end{cases}
$$

Solution. The Fourier series of $f(x)$ is given by

$$
f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx).
$$

We find the Fourier coefficients. By the definition,

$$\begin{aligned}
a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx \\
&= \frac{1}{\pi}\left[\int_{-\pi}^{0} -x\,dx + \int_{0}^{\pi} x\,dx\right] \\
&= \frac{1}{\pi}\left[\left.\frac{-x^2}{2}\right|_{-\pi}^{0} + \left.\frac{x^2}{2}\right|_{0}^{\pi}\right] \\
&= \frac{1}{\pi}\left[0 + \frac{(-\pi)^2}{2}\frac{\pi^2}{2} - 0\right] \\
&= \frac{1}{\pi}(\pi^2) \\
&= \pi.
\end{aligned}$$

For $n = 1, 2, \ldots,$

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \\
&= \frac{1}{\pi}\left[\int_{-\pi}^{0}(-x)\cos nx\,dx + \int_{0}^{\pi} x\cos nx\,dx\right] \\
&= \frac{1}{\pi}\left[\left.(-x)\frac{\sin nx}{n}\right|_{-\pi}^{0} - \int_{-\pi}^{0}(-1)\frac{\sin nx}{n}\,dx + \left.x\frac{\sin nx}{n}\right|_{0}^{\pi} - \int_{0}^{\pi}(1)\frac{\sin nx}{n}\,dx\right] \\
&= \frac{1}{\pi}\left[0 + \int_{-\pi}^{0}\frac{\sin nx}{n}\,dx + 0 - \int_{0}^{\pi}\frac{\sin nx}{n}\,dx\right] \\
&= \frac{1}{\pi}\left[\int_{-\pi}^{0}\frac{\sin nx}{n}\,dx - \int_{0}^{\pi}\frac{\sin nx}{n}\,dx\right] \\
&= \frac{1}{\pi}\left[\left.\frac{-\cos nx}{n^2}\right|_{-\pi}^{0} - \left.\frac{-\cos nx}{n^2}\right|_{0}^{\pi}\right] \\
&= \frac{1}{\pi}\left[0 + \left(\frac{\cos n\pi}{n^2} - \frac{\cos 0}{n^2}\right)\right] \\
&= \frac{1}{n^2\pi}[(-1)^n - 1] \\
&= \frac{1}{n^2\pi}\begin{cases} -2, & n = 1, 3, 5, \ldots \\ 0, & n = 2, 4, 6, \ldots. \end{cases}
\end{aligned}$$

and

$$\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \\
&= \frac{1}{\pi}\left[\int_{-\pi}^{0}(-x)\sin nx\,dx + \int_{0}^{\pi} x\sin nx\,dx\right]
\end{aligned}$$

In the first integral substitute $u = -x$, then $-x = u, \sin nx = -\sin nu, dx = -du, x = -\pi \Rightarrow u = \pi$, and $x = 0 \Rightarrow u = 0$, so that

$$\begin{aligned}
&= \frac{1}{\pi}\left[\int_{\pi}^{0} u \sin nu\, du + \int_{0}^{\pi} x \sin nx\, dx\right] \\
&= \frac{1}{\pi}\left[-\int_{0}^{\pi} x \sin nx\, dx + \int_{0}^{\pi} x \sin nx\, dx\right] \\
&= 0
\end{aligned}$$

Hence

$$\begin{aligned}
f(x) &= \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + \cdots + b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + \cdots \\
&= \frac{\pi}{4} - \frac{2}{\pi}\cos x + 0 - \frac{2}{3^2\pi}\cos 3x + 0 - \frac{2}{5^2\pi}\cos 5x + \cdots + 0 \\
&= \frac{\pi}{4} - \frac{2}{\pi}\left(\frac{\cos x}{1} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right).
\end{aligned}$$

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Problem 2.6

Find the Fourier series of

$$f(x) = 1 + x, \quad -\pi < x < \pi.$$

Solution. The Fourier series of $f(x)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx).$$

We find the Fourier coefficients. By the definition,

$$\begin{aligned}
a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} (1 + x)\, dx \\
&= \frac{1}{\pi}\left[\int_{-\pi}^{\pi} 1\, dx + \int_{-\pi}^{\pi} x\, dx\right] \\
&= \frac{1}{\pi}\left[[x]_{-\pi}^{\pi} + 0\right] \\
&= \frac{1}{\pi}(\pi - (-\pi)) \\
&= \frac{1}{\pi}2\pi \\
&= 2.
\end{aligned}$$

For $n = 1, 2, \ldots,$

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} (1+x)\cos nx\,dx \\
&= \frac{1}{\pi}\left[\int_{-\pi}^{\pi}\cos nx\,dx + \int_{-\pi}^{\pi} x\cos nx\,dx\right] \\
&= \frac{1}{\pi}\left[2\int_{0}^{\pi}\cos nx\,dx + 0\right] \\
&= \frac{1}{\pi}\left[2\frac{\sin nx}{n}\right]_0^{\pi} \\
&= \frac{2}{\pi}\left[\frac{\sin n\pi}{n} - \frac{\sin 0}{n}\right] \\
&= 0.
\end{aligned}$$

and

$$\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} (1+x)\sin nx\,dx \\
&= \frac{1}{\pi}\left[\int_{-\pi}^{\pi}\sin nx\,dx + \int_{-\pi}^{\pi} x\sin nx\,dx\right] \\
&= \frac{1}{\pi}\left[0 + 2\int_{0}^{\pi} x\sin nx\,dx\right] \\
&= \frac{2}{\pi}\left[x\frac{-\cos nx}{n}\Big|_0^{\pi} - \int_0^{\pi}\frac{-\cos nx}{n}\,dx\right] \\
&= \frac{2}{\pi}\left[\pi\frac{-\cos n\pi}{n} - 0 + \frac{\sin nx}{n^2}\Big|_0^{\pi}\right] \\
&= \frac{2}{\pi}\left[\pi\frac{-(-1)^n}{n} + \frac{\sin n\pi}{n^2} - \frac{\sin 0}{n^2}\right] \\
&= \frac{2}{\pi}\left[\pi\frac{(-1)^{n+1}}{n} + 0 - 0\right] \\
&= \frac{2}{\pi}\pi\frac{(-1)^{n+1}}{n} \\
&= \frac{2(-1)^{n+1}}{n}.
\end{aligned}$$

Hence

$$\begin{aligned}
f(x) &= \frac{a_0}{2} + a_1\cos x + a_2\cos 2x + a_3\cos 3x + \cdots + b_1\sin x + b_2\sin 2x + b_3\sin 3x + \cdots \\
&= 1 + 0 + 2\sin x - 2\frac{\sin 2x}{2} + 2\frac{\sin 3x}{3} - 2\frac{\sin 4x}{4} + \cdots \\
&= 1 + 2\left(\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \cdots\right).
\end{aligned}$$

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Problem 2.7

Find the Fourier series of

$$f(x) = \begin{cases} 0, & -\pi < x < 0 \\ \sin x, & 0 < x < \pi. \end{cases}$$

Solution. The Fourier series of $f(x)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx).$$

We find the Fourier coefficients. By the definition,

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx \\ &= \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\,dx + \int_{0}^{\pi} \sin x\,dx\right] \\ &= \frac{1}{\pi}\left[0 + -\cos x|_0^\pi\right] \\ &= \frac{1}{\pi}\left[-\cos\pi - (-\cos 0)\right] \\ &= \frac{1}{\pi}(1+1) \\ &= \frac{2}{\pi}. \end{aligned}$$

For $n = 1, 2, \ldots,$

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \\ &= \frac{1}{\pi}\left[\int_{-\pi}^{0} (0)\cos nx\,dx + \int_{0}^{\pi} \sin x \cos nx\,dx\right] \\ &= \frac{1}{\pi}\int_{0}^{\pi} \sin x \cos nx\,dx. \end{aligned}$$

Let $I_1 = \int_0^\pi \sin x \cos nx\, dx$. Then

$$\begin{aligned}
I_1 &= \int_0^\pi \sin x \cos nx\, dx \\
&= \left[\sin x \frac{\sin nx}{n}\right]_0^\pi - \int_0^\pi \cos x \frac{\sin nx}{n}\, dx \\
&= \sin\pi \frac{\sin n\pi}{n} - \sin 0 \frac{\sin 0}{n} - \left[\cos x \frac{-\cos nx}{n^2}\right]_0^\pi + \int_0^\pi (-\sin x)\frac{-\cos nx}{n^2}\, dx \\
&= 0 - 0 - \left[\cos\pi \frac{-\cos n\pi}{n^2} - \cos 0 \frac{-\cos 0}{n^2}\right] + \frac{1}{n^2}\int_0^\pi \sin x \cos nx\, dx \\
&= -\left[(-1)\frac{-(-1)^n}{n^2} - (1)\frac{-1}{n^2}\right] + \frac{1}{n^2} I_1
\end{aligned}$$

$$\Rightarrow \left(1 - \frac{1}{n^2}\right) I_1 = \frac{(-1)^{n+1}}{n^2} - \frac{1}{n^2}$$

$$\Rightarrow \left(\frac{n^2-1}{n^2}\right) I_1 = \frac{(-1)^{n+1} - 1}{n^2}$$

$$\Rightarrow I_1 = \frac{(-1)^{n+1} - 1}{n^2 - 1} \quad (\text{for } n \neq 1).$$

When $n = 1$,

$$\begin{aligned}
a_1 &= \frac{1}{\pi}\int_0^\pi \sin x \cos x\, dx. \\
&= \frac{1}{\pi}\int_0^\pi \frac{\sin 2x}{2}\, dx \\
&= \frac{1}{\pi}\left[\frac{-\cos 2x}{4}\right]_0^\pi \\
&= -\frac{1}{4\pi}[\cos 2\pi - \cos 0] \\
&= -\frac{1}{4\pi}[1 - 1] \\
&= 0.
\end{aligned}$$

Hence $a_1 = 0$ and for $n > 1$,

$$\begin{aligned}
a_n &= \frac{1}{\pi}\frac{(-1)^{n+1} - 1}{n^2 - 1} \\
&= \begin{cases} 0 & \text{if } n \text{ is odd} \\ \frac{-2}{\pi(n^2-1)} & \text{if } n \text{ is even} \end{cases}
\end{aligned}$$

Also

$$\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx \\
&= \frac{1}{\pi}\left[\int_{-\pi}^{0} (0)\sin nx\, dx + \int_0^\pi \sin x \sin nx\, dx\right] \\
&= \frac{1}{\pi}\int_0^\pi \sin x \sin nx\, dx.
\end{aligned}$$

Let $I_2 = \int_0^\pi \sin x \sin nx\, dx$. Then

$$\begin{aligned}
I_2 &= \int_0^\pi \sin x \sin nx\, dx \\
&= \left[\sin x \frac{-\cos nx}{n}\right]_0^\pi - \int_0^\pi \cos x \frac{-\cos nx}{n}\, dx \\
&= \sin \pi \frac{-\cos n\pi}{n} - \sin 0 \frac{-\cos 0}{n} + \left[\cos x \frac{\sin nx}{n^2}\right]_0^\pi - \int_0^\pi (-\sin x) \frac{-\sin nx}{n^2}\, dx \\
&= 0 - 0 + \left[\cos \pi \frac{\sin n\pi}{n^2} - \cos 0 \frac{\sin 0}{n^2}\right] + \frac{1}{n^2} \int_0^\pi \sin x \sin nx\, dx \\
&= 0 + \frac{1}{n^2} I_2
\end{aligned}$$

$$\Rightarrow \left(1 - \frac{1}{n^2}\right) I_2 = 0$$

$$\Rightarrow \left(\frac{n^2 - 1}{n^2}\right) I_2 = 0$$

$$\begin{aligned}
\Rightarrow I_2 &= \frac{0}{n^2 - 1} \\
&= 0. \quad (\text{for } n \neq 1)
\end{aligned}$$

If $n = 1$,

$$\begin{aligned}
b_1 &= \frac{1}{\pi} \int_0^\pi \sin x \sin x\, dx \\
&= \frac{1}{\pi} \int_0^\pi \sin^2 x\, dx \\
&= \frac{1}{\pi} \int_0^\pi \frac{1 - \cos 2x}{2}\, dx \\
&= \frac{1}{2\pi} \left[x - \frac{\sin 2x}{2}\right]_0^\pi \\
&= \frac{1}{2\pi} \left[(\pi - 0) - \frac{\sin 2\pi}{2} + \frac{\sin 0}{2}\right] \\
&= \frac{1}{2\pi} [\pi - 0 + 0] \\
&= \frac{1}{2}.
\end{aligned}$$

Hence

$$\begin{aligned}
f(x) &= \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + \cdots + b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + \cdots \\
&= \frac{1}{\pi} + 0 - \frac{2}{\pi(2^2 - 1)} \cos 2x + 0 - \frac{2}{\pi(4^2 - 1)} \cos 4x + 0 - \frac{2}{\pi(6^2 - 1)} \cos 6x + \cdots \\
&\quad + \frac{1}{2} \sin x + 0 + 0 + \cdots \\
&= \frac{1}{\pi} + \frac{\sin x}{2} - \frac{2}{\pi} \left(\frac{\cos 2x}{2^2 - 1} + \frac{\cos 4x}{4^2 - 1} + \frac{\cos 6x}{6^2 - 1} + \cdots\right).
\end{aligned}$$

■

We have discussed Fourier series of periodic functions in the interval $[-\pi, \pi]$. Now we turn our discussion to the Fourier series of functions in an interval $[-l, l]$. Suppose that $f(x)$ is a given function of period $2l$ defined in the interval $[-l, l]$. The *Fourier series* of $f(x)$ is defined as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right) \tag{6}$$

where the Fourier coefficients a_0, a_n and b_n are given by

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx \tag{7}$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx \tag{8}$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx \tag{9}$$

for $n = 1, 2, \ldots$.

▶ For the basic interval $(0, 2l)$ we need only change the integration limits to 0 to $2l$.

Problem 2.8

Find the Fourier series of

$$f(x) = \begin{cases} 0, & 0 < x < l \\ 1, & l < x < 2l. \end{cases}$$

Solution. The Fourier series of $f(x)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right).$$

We find the Fourier coefficients. By the definition,

$$\begin{aligned} a_0 &= \frac{1}{l}\int_0^{2l} f(x)\,dx \\ &= \frac{1}{l}\left[\int_0^{l} 0\,dx + \int_l^{2l} 1\,dx\right] \\ &= \frac{1}{l}\int_l^{2l} 1\,dx \\ &= \frac{1}{l}[x]_l^{2l} \\ &= \frac{1}{l}[2l - l] \\ &= 1. \end{aligned}$$

For $n = 1, 2, \ldots,$

$$\begin{aligned}
a_n &= \frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi x}{l}\,dx \\
&= \frac{1}{l}\left[\int_0^{l}(0)\cos\frac{n\pi x}{l}\,dx + \int_l^{2l}(1)\cos\frac{n\pi x}{l}\,dx\right] \\
&= \frac{1}{l}\int_l^{2l}\cos\frac{n\pi x}{l}\,dx \\
&= \frac{1}{l}\left[\frac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right]_l^{2l} \\
&= \frac{1}{l}\frac{l}{n\pi}\left[\sin\frac{n\pi x}{l}\right]_l^{2l} \\
&= \frac{1}{n\pi}\left[\sin\frac{2ln\pi}{l} - \sin\frac{ln\pi}{l}\right] \\
&= \frac{1}{n\pi}[\sin 2n\pi - \sin n\pi] \\
&= 0.
\end{aligned}$$

Also

$$\begin{aligned}
b_n &= \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}\,dx \\
&= \frac{1}{l}\left[\int_0^{l}(0)\sin\frac{n\pi x}{l}\,dx + \int_l^{2l}(1)\sin\frac{n\pi x}{l}\,dx\right] \\
&= \frac{1}{l}\int_l^{2l}\sin\frac{n\pi x}{l}\,dx \\
&= \frac{1}{l}\left[\frac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right]_l^{2l} \\
&= \frac{1}{l}\frac{l}{n\pi}\left[-\cos\frac{n\pi x}{l}\right]_l^{2l} \\
&= \frac{-1}{n\pi}\left[\cos\frac{2ln\pi}{l} - \cos\frac{ln\pi}{l}\right] \\
&= \frac{-1}{n\pi}[\cos 2n\pi - \cos n\pi] \\
&= \frac{-1}{n\pi}[1-(-1)^n] \\
&= \frac{1}{n\pi}\begin{cases} -2 & n = 1, 3, 5, \ldots \\ 0 & n = 2, 4, 6, \ldots. \end{cases}
\end{aligned}$$

Hence

$$\begin{aligned}
f(x) &= \frac{a_0}{2} + a_1\cos\frac{\pi x}{l} + a_2\cos\frac{2\pi x}{l} + a_3\cos\frac{3\pi x}{l} + \cdots + b_1\sin\frac{\pi x}{l} + b_2\sin\frac{2\pi x}{l} + b_3\sin\frac{3\pi x}{l} + \cdots \\
&= \frac{1}{2} + 0 + \frac{1}{\pi}\left(\frac{-2}{1}\sin\frac{\pi x}{l} + 0 + \frac{-2}{3}\sin\frac{3\pi x}{l} + 0 + \frac{-2}{5}\sin\frac{5\pi x}{l} + \cdots\right) \\
&= \frac{1}{2} - \frac{2}{\pi}\left(\sin\frac{\pi x}{l} + \frac{1}{3}\sin\frac{3\pi x}{l} + \frac{1}{5}\sin\frac{5\pi x}{l} + \cdots\right).
\end{aligned}$$

■

Problem 2.9

A sinusoidal voltage $E\sin\omega t$, where t is time, is passed through a half-wave rectifier that clips the negative portion of the wave. Find the Fourier series of the resulting periodic function

$$u(t) = \begin{cases} 0 & \text{if } -L < t < 0 \\ E\sin\omega t & \text{if } 0 < t < L. \end{cases}$$

Period, $p = 2L = \frac{2\pi}{\omega}$.

Solution. We have $L = \frac{\pi}{\omega}$. The Fourier series of $f(x)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi t}{L} + b_n \sin\frac{n\pi t}{L}\right) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos n\omega t + b_n \sin n\omega t).$$

We find the Fourier coefficients. By the definition,

$$\begin{aligned}
a_0 &= \frac{1}{L}\int_{-L}^{L} f(t)\,dt \\
&= \frac{1}{L}\left[\int_{-L}^{0} 0\,dt + \int_{0}^{2L} E\sin\omega t\,dt\right] \\
&= \frac{1}{L}\int_{0}^{L} E\sin\omega t\,dt \\
&= \frac{E}{L}\left[\frac{-\cos\omega t}{\omega}\right]_0^L \\
&= -\frac{E}{L\omega}[\cos L\omega - \cos 0] \\
&= -\frac{E}{\pi}[\cos\pi - 1] \\
&= -\frac{E}{\pi}[-1-1] \\
&= \frac{2E}{\pi}.
\end{aligned}$$

For $n = 1, 2, \ldots,$

$$\begin{aligned}
a_n &= \frac{1}{L}\int_{-L}^{L} f(t)\cos\frac{n\pi t}{L}\,dt \\
&= \frac{1}{L}\int_{-L}^{L} f(t)\cos n\omega t\,dt \\
&= \frac{1}{L}\left[\int_{-L}^{0}(0)\cos n\omega t\,dt + \int_{0}^{L} E\sin\omega t\cos n\omega t\,dt\right] \\
&= \frac{1}{L}\int_{0}^{L} E\sin\omega t\cos n\omega t\,dt \\
&= \frac{E}{L}\int_{0}^{L}\frac{1}{2}[\sin(1+n)\omega t + \sin(1-n)\omega t]\,dt
\end{aligned}$$

For $n \neq 1$, we get

$$\begin{aligned}
a_n &= \frac{E}{2L}\left[\frac{-\cos(1+n)\omega t}{(1+n)\omega} + \frac{-\cos(1-n)\omega t}{(1-n)\omega}\right]_0^L \\
&= \frac{E}{2L\omega}\left[\frac{-\cos(1+n)L\omega + \cos 0}{1+n} + \frac{-\cos(1-n)\omega L + \cos 0}{1-n}\right] \\
&= \frac{E}{2\pi}\left[\frac{-\cos(1+n)\pi + 1}{1+n} + \frac{-\cos(1-n)\pi + 1}{1-n}\right] \\
&= \frac{E}{2\pi}\left[\frac{-(-1)^{1+n} + 1}{1+n} + \frac{-(-1)^{1-n} + 1}{1-n}\right] \\
&= \frac{E}{2\pi}\left[\frac{(-1)^{2+n} + 1}{1+n} + \frac{(-1)^{2-n} + 1}{1-n}\right] \\
&= \frac{E}{2\pi}\begin{cases} 0 & n = 3, 5, \ldots \\ \frac{2}{1+n} + \frac{2}{1-n} & n = 2, 4, 6, \ldots \end{cases} \\
&= \begin{cases} 0 & n = 3, 5, \ldots \\ \frac{2E}{(1+n)(1-n)\pi} & n = 2, 4, 6, \ldots . \end{cases}
\end{aligned}$$

For $n = 1$,

$$\begin{aligned}
a_1 &= \frac{E}{L}\int_0^L \frac{1}{2}[\sin 2\omega t + \sin 0]\, dt \\
&= \frac{E}{2L}\int_0^L \sin 2\omega t\, dt \\
&= \frac{E}{2L}\left[\frac{-\cos 2\omega t}{2\omega}\right]_0^L \\
&= \frac{-E}{4L\omega}[\cos 2\omega L - \cos 0] \\
&= \frac{-E}{4L\omega}[\cos 2\pi - \cos 0] \\
&= \frac{-E}{4L\omega}[1 - 1] \\
&= 0.
\end{aligned}$$

Also

$$\begin{aligned}
b_n &= \frac{1}{L}\int_{-L}^L f(t)\sin\frac{n\pi t}{L}\, dt \\
&= \frac{1}{L}\int_{-L}^L f(t)\sin n\omega t\, dt \\
&= \frac{1}{L}\left[\int_{-L}^0 (0)\sin n\omega t\, dt + \int_0^L E\sin\omega t \sin n\omega t\, dt\right] \\
&= \frac{1}{L}\int_0^L E\sin\omega t\sin n\omega t\, dt \\
&= \frac{E}{L}\int_0^L \frac{1}{2}[\cos(1-n)\omega t - \cos(1+n)\omega t]\, dt
\end{aligned}$$

For $n \neq 1$, we get

$$\begin{aligned}
b_n &= \frac{E}{2L}\left[\frac{\sin(1-n)\omega t}{(1-n)\omega} - \frac{\sin(1+n)\omega t}{(1+n)\omega}\right]_0^L \\
&= \frac{E}{2L\omega}\left[\frac{\sin(1-n)L\omega - \sin 0}{1-n} - \frac{\sin(1+n)\omega L - \sin 0}{1-n}\right] \\
&= \frac{E}{2\pi}\left[\frac{\sin(1-n)\pi}{1-n} - \frac{\sin(1+n)\pi}{1+n}\right] \\
&= \frac{E}{2\pi}[0-0] \\
&= 0.
\end{aligned}$$

For $n = 1$,

$$\begin{aligned}
b_1 &= \frac{E}{L}\int_0^L \frac{1}{2}[\cos 0 - \cos 2\omega t]\,dt \\
&= \frac{E}{2L}\int_0^L [1 - \cos 2\omega t]\,dt \\
&= \frac{E}{2L}\left[t - \frac{\sin 2\omega t}{2\omega}\right]_0^L \\
&= \frac{E}{2L}\left[L - 0 - \frac{\sin 2L\omega - \sin 0}{2\omega}\right] \\
&= \frac{E}{2L}\left[L - \frac{\sin 2\pi}{2\omega}\right] \\
&= \frac{E}{2L}[L-0] \\
&= \frac{E}{2}.
\end{aligned}$$

Hence

$$\begin{aligned}
u(t) &= \frac{a_0}{2} + a_1\cos\omega t + a_2\cos 3\omega t + a_3\cos 3\omega t + \cdots + b_1\sin\omega t + b_2\sin 2\omega t + b_3\sin 3\omega t + \cdots \\
&= \frac{E}{\pi} + 0 + \frac{2E}{3(-1)\pi}\cos 2\omega t + 0 + \frac{2E}{5(-3)\pi}\cos 4\omega t + \cdots + \frac{E}{2}\sin\omega t + 0 \\
&= \frac{E}{\pi} - \frac{2E}{\pi}\left(\frac{1}{1\cdot 3}\cos 2\omega t + \frac{1}{3\cdot 5}\cos\omega 4t + \cdots\right) + \frac{E}{2}\sin\omega t.
\end{aligned}$$

■

Problem 2.10

Find the Fourier series of

$$f(x) = \begin{cases} 1+2x, & -1 < x < 0 \\ 1-2x, & 0 < x < 1. \end{cases}$$

Solution. Here $l = 1$. The Fourier series of $f(x)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{l} + b_n\sin\frac{n\pi x}{l}\right) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos n\pi x + b_n\sin n\pi x).$$

We find the Fourier coefficients. By the definition,

$$\begin{aligned} a_0 &= \frac{1}{l}\int_{-l}^{l} f(x)\,dx \\ &= \int_{-1}^{1} f(x)\,dx \\ &= \int_{-1}^{0} 1+2x\,dx + \int_{0}^{1} 1-2x\,dx \\ &= [x+x^2]_{-1}^{0} + [x-x^2]_0^1 \\ &= [0+0-(-1+(-1)^2)] + [1-1^2-(0-0)] \\ &= 0. \end{aligned}$$

For $n = 1, 2, \ldots$,

$$\begin{aligned} a_n &= \int_{-1}^{1} f(x)\cos n\pi x\,dx \\ &= \int_{-1}^{0}(1+2x)\cos n\pi x\,dx + \int_0^1 (1-2x)\cos n\pi x\,dx] \\ &= \ (1+2x)\frac{\sin n\pi x}{n\pi}\Big|_{-1}^{0} - \int_{-1}^{0}(2)\frac{\sin n\pi x}{n\pi}\,dx + \ (1-2x)\frac{\sin n\pi x}{n\pi}\Big|_{0}^{1} - \int_0^1(-2)\frac{\sin n\pi x}{n\pi}\,dx \\ &= 0 + \frac{2}{n\pi}\ \frac{\cos n\pi x}{n\pi}\Big|_{-1}^{0} + 0 - \frac{2}{n\pi}\ \frac{\cos n\pi x}{n\pi}\Big|_{0}^{1} \\ &= \frac{2}{n^2\pi^2}[\cos 0 - \cos(-n\pi)] - \frac{4}{n^2\pi^2}[\cos n\pi - \cos 0] \\ &= \frac{2}{n^2\pi^2}[1-(-1)^n] - \frac{4}{n^2\pi^2}[(-1)^n - 1] \\ &= \frac{2}{n^2\pi^2}[2-2(-1)^n] \\ &= \frac{4}{n^2\pi^2}[1-(-1)^n] \\ &= \frac{4}{n^2\pi^2}\begin{cases} 2 & n = 1,3,5,\ldots \\ 0 & n = 2,4,6,\ldots \end{cases} \\ &= \frac{8}{\pi^2}\begin{cases} \frac{1}{n^2} & n = 1,3,5,\ldots \\ 0 & n = 2,4,6,\ldots. \end{cases} \end{aligned}$$

Also

$$
\begin{aligned}
b_n &= \int_{-1}^{1} f(x)\sin n\pi x\,dx \\
&= \int_{-1}^{0}(1+2x)\sin n\pi x\,dx + \int_{0}^{1}(1-2x)\sin n\pi x\,dx] \\
&= (1+2x)\frac{-\cos n\pi x}{n\pi}\Big|_{-1}^{0} - \int_{-1}^{0}(2)\frac{-\cos n\pi x}{n\pi}\,dx + (1-2x)\frac{-\cos n\pi x}{n\pi}\Big|_{0}^{1} - \int_{0}^{1}(-2)\frac{-\cos n\pi x}{n\pi}\,dx \\
&= \frac{-1}{n\pi}[(1+0)\cos 0 - (1-2)\cos(-n\pi)] + \frac{2}{n\pi}\left[\frac{\sin n\pi x}{n\pi}\right]_{-1}^{0} + \\
&\quad \frac{-1}{n\pi}[(1-2)\cos n\pi - (1-0)\cos 0] - \frac{2}{n\pi}\left[\frac{\sin n\pi x}{n\pi}\right]_{0}^{1} \\
&= \frac{-1}{n\pi}[1+(-1)^n] + 0 - \frac{1}{n\pi}[-(-1)^n - 1] - 0 \\
&= \frac{-1}{n\pi}[1+(-1)^n - (-1)^n - 1] \\
&= 0.
\end{aligned}
$$

Hence

$$
\begin{aligned}
f(x) &= \frac{a_0}{2} + a_1\cos\pi x + a_2\cos 2\pi x + a_3\cos 3\pi x + \cdots + b_1\sin\pi x + b_2\sin 2\pi x + b_3\sin 3\pi x + \cdots \\
&= 0 + \frac{8}{\pi^2}\left(\frac{1}{1^2}\cos\pi x + 0 + \frac{1}{3^2}\cos 3\pi x + 0 + \frac{1}{5^2}\cos 5\pi x + \cdots\right) + 0 \\
&= \frac{8}{\pi^2}\left(\frac{\cos\pi x}{1^2} + \frac{\cos 3\pi x}{3^2} + \frac{\cos 5\pi x}{5^2} + \cdots\right).
\end{aligned}
$$

■

Problem 2.11

Find the Fourier series of

$$
f(x) = \begin{cases} 0, & -1 < x < 0 \\ 1, & 0 < x < 3. \end{cases}
$$

Solution. Here $2L = 4$ so that $L = 2$. Left as exercise. ■

Problem 2.12

Find the Fourier series of

$$
f(x) = \begin{cases} 0, & -\frac{1}{2} < x < 0 \\ x, & 0 < x < \frac{1}{2}. \end{cases}
$$

Solution. Here $2L = 1$ so that $L = \frac{1}{2}$. Left as exercise. ■

Problem 2.13

Find the Fourier series of

$$f(x) = \begin{cases} \frac{x}{2}, & 0 < x < 2 \\ 1, & 2 < x < 3. \end{cases}$$

Solution. Here $2L = 3$ so that $L = \frac{3}{2}$. Left as exercise. ■

EVEN AND ODD FUNCTIONS

We know that a function is *even* if $f(-x) = f(x)$ and *odd* if $f(-x) = -f(x)$. It is clear that product of two even/odd functions is even, but a product of two functions of different category will be an odd function. It is also known that

$$\int_{-l}^{l} f(x)\,dx = \begin{cases} 2\int_0^l f(x)\,dx & \text{if } f(x) \text{ is even} \\ 0 & \text{if } f(x) \text{ is odd.} \end{cases}$$

Using these facts about even/odd functions, we can deduce that

$$\text{If } f(x) \text{ is odd, } \begin{cases} a_n = 0, \\ b_n = \frac{2}{l}\int_0^l f(x) \sin\frac{n\pi x}{l}\,dx. \end{cases} \tag{10}$$

and

$$\text{If } f(x) \text{ is even, } \begin{cases} a_n = \frac{2}{l}\int_0^l f(x) \cos\frac{n\pi x}{l}\,dx, \\ b_n = 0. \end{cases} \tag{11}$$

We say that we have expanded an odd function $f(x)$ in a sine series

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

and an even function $f(x)$ in a cosine series

$$f(x) = \sum_{n=0}^{\infty} a_n \sin\frac{n\pi x}{l}.$$

3 CONVERGENCE AND SUM OF A FOURIER SERIES

We next give the conditions under which a function can be expanded as a Fourier series.

Theorem 3.1

Consider a function $f(x)$ defined in the interval $[-\pi, \pi]$ satisfying the following conditions;

- *$f(x)$ is periodic with period 2π.*
- *$f(x)$ is piecewise continuous in the interval $[-\pi, \pi]$*
- *$f(x)$ has left-hand and right-hand derivatives at each point of $[-\pi, \pi]$.*

Then the Fourier series of $f(x)$ converges. It sum is $f(x)$ at points of continuity. If $f(x)$ is discontinuous at x_0 in $[-\pi, \pi]$, then at that point sum of the series is the average of the left-hand and right-hand limits of $f(x)$ at x_0.

Note

- The left hand limit of $f(x)$ at x_0 is defined as $f(x_0^-) = \lim_{h \to 0} f(x_0 - h)$ where $h \to 0$ through positive values.
- The right hand limit of $f(x)$ at x_0 is defined as $f(x_0^+) = \lim_{h \to 0} f(x_0 + h)$ where $h \to 0$ through positive values.
- The left hand derivative of $f(x)$ at x_0 is defined as $f'(x_0^-) = \lim_{h \to 0} \frac{f(x_0 - h) - f(x_0^-)}{-h}$ where $h \to 0$ through positive values.
- The right hand derivative of $f(x)$ at x_0 is defined as $f'(x_0^+) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0^+)}{-h}$ where $h \to 0$ through positive values.

Problem 3.1

Find the Fourier series of the periodic function $f(x)$ defined by

$$f(x) = \begin{cases} -k & \text{if } -\pi < x < 0 \\ k & \text{if } 0 < x < \pi \end{cases} \quad \text{and } f(x+2\pi) = f(x).$$

Using it deduce that $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}$.

Solution. It can be seen that the Fourier series of $f(x)$ is

$$f(x) = \frac{4k}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\right).$$

Since the function $f(x)$ is continuous at $x = \frac{\pi}{2}$ and $f\left(\frac{\pi}{2}\right) = k$, we see that

$$\begin{aligned} k &= \frac{4k}{\pi}\left(\sin\frac{\pi}{2} + \frac{1}{3}\sin\frac{3\pi}{2} + \frac{1}{5}\sin\frac{5\pi}{2} + \frac{1}{7}\sin\frac{7\pi}{2}\cdots\right) \\ &= \frac{4k}{\pi}\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right) \\ \Rightarrow \frac{\pi}{4} &= 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots. \end{aligned}$$

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Problem 3.2

Find the Fourier series of

$$f(x) = x^2, \quad -1 < x < 1, \quad p = 2.$$

Using it deduce that $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} = \frac{\pi^2}{6}$.

Solution. Here $l = 1$. The Fourier series of $f(x)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{l} + b_n\sin\frac{n\pi x}{l}\right) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos n\pi x + b_n\sin n\pi x).$$

We find the Fourier coefficients. By the definition,

$$\begin{aligned}
a_0 &= \frac{1}{l}\int_{-l}^{l} f(x)\,dx \\
&= \int_{-1}^{1} f(x)\,dx \\
&= \int_{-1}^{1} x^2\,dx \\
&= 2\int_{0}^{1} x^2\,dx \\
&= 2\left[\frac{x^3}{3}\right]_0^1 \\
&= 2\left[\frac{1}{3}-0\right] \\
&= \frac{2}{3}.
\end{aligned}$$

For $n = 1, 2, \ldots,$

$$\begin{aligned}
a_n &= \int_{-1}^{1} f(x)\cos n\pi x\,dx \\
&= \int_{-1}^{1} x^2 \cos n\pi x\,dx \\
&= 2\int_{0}^{1} x^2 \cos n\pi x\,dx \\
&= 2\left[x^2 \frac{\sin n\pi x}{n\pi}\bigg|_0^1 - \int_0^1 2x\frac{\sin n\pi x}{n\pi}\,dx\right] \\
&= \frac{2}{n\pi}\left[x^2 \sin n\pi x|_0^1 - 2\int_0^1 x\sin n\pi x\,dx\right] \\
&= \frac{2}{n\pi}\left[\sin n\pi - 0 - 2\left(x\frac{-\cos n\pi x}{n\pi}\bigg|_0^1 - \int_0^1 \frac{-\cos n\pi x}{n\pi}\,dx\right)\right] \\
&= \frac{2}{n^2\pi^2}\left[-2\left(-x\cos n\pi x|_0^1 + \int_0^1 \cos n\pi x\,dx\right)\right] \\
&= \frac{-4}{n^2\pi^2}\left[-\cos n\pi - 0 + \frac{\sin n\pi x}{n\pi}\bigg|_0^1\right] \\
&= \frac{-4}{n^2\pi^2}\left[(-1)^{n+1} + \frac{\sin n\pi}{n\pi} - \frac{\sin 0}{n\pi}\right] \\
&= \frac{-4}{n^2\pi^2}\left[(-1)^{n+1} + 0 - 0\right] \\
&= \frac{4(-1)^n}{n^2\pi^2}.
\end{aligned}$$

Also

$$\begin{aligned} b_n &= \int_{-1}^{1} f(x)\sin n\pi x\,dx \\ &= \int_{-1}^{1} x^2 \sin n\pi x\,dx \\ &= 0. \end{aligned}$$

Hence

$$\begin{aligned} f(x) &= \frac{a_0}{2} + a_1\cos\pi x + a_2\cos 2\pi x + a_3\cos 3\pi x + \cdots + b_1\sin\pi x + b_2\sin 2\pi x + b_3\sin 3\pi x + \cdots \\ &= \frac{1}{3} + \frac{4}{\pi^2}\left(-\frac{1}{1^2}\cos\pi x + \frac{1}{2^2}\cos 2\pi x - \frac{1}{3^2}\cos 3\pi x + \frac{1}{4^2}\cos 4\pi x - \frac{1}{5^2}\cos 5\pi x + \cdots\right) + 0 \\ &= \frac{1}{3} + \frac{4}{\pi^2}\left(-\frac{1}{1^2}\cos\pi x + \frac{1}{2^2}\cos 2\pi x - \frac{1}{3^2}\cos 3\pi x + \frac{1}{4^2}\cos 4\pi x - \frac{1}{5^2}\cos 5\pi x + \cdots\right) \end{aligned}$$

Put $x = 1$. We have $f(1^-) = 1$ so that

$$\begin{aligned} 1 &= \frac{1}{3} + \frac{4}{\pi^2}\left(-\frac{1}{1^2}\cos\pi + \frac{1}{2^2}\cos 2\pi - \frac{1}{3^2}\cos 3\pi + \frac{1}{4^2}\cos 4\pi - \frac{1}{5^2}\cos 5\pi + \cdots\right) \\ \Rightarrow 1 - \frac{1}{3} &= \frac{4}{\pi^2}\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots\right) \\ \Rightarrow \frac{2}{3} &= \frac{4}{\pi^2}\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots\right) \\ \Rightarrow \frac{2\pi^2}{12} &= \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots \\ \Rightarrow \frac{\pi^2}{6} &= \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots . \end{aligned}$$

Put $x = \frac{1}{2}$. We have $f\left(\frac{1}{2}\right) = \frac{1}{4}$ so that

$$\begin{aligned} \frac{1}{4} &= \frac{1}{3} + \frac{4}{\pi^2}\left(-\frac{1}{1^2}\cos\frac{\pi}{2} + \frac{1}{2^2}\cos 2\frac{\pi}{2} - \frac{1}{3^2}\cos\frac{3\pi}{2} + \frac{1}{4^2}\cos 4\frac{\pi}{2} - \frac{1}{5^2}\cos\frac{5\pi}{2} + \cdots\right) \\ \Rightarrow \frac{1}{4} - \frac{1}{3} &= \frac{4}{\pi^2}\left(-\frac{1}{2^2} + \frac{1}{4^2} - \frac{1}{6^2} + \cdots\right) \\ \Rightarrow \frac{-1}{12} &= \frac{4}{\pi^2}\left(-\frac{1}{2^2} + \frac{1}{4^2} - \frac{1}{6^2} + \cdots\right) \\ \Rightarrow \frac{\pi^2}{48} &= \frac{1}{2^2} - \frac{1}{4^2} + \frac{1}{6^2} - \frac{1}{8^2} + \cdots . \end{aligned}$$

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